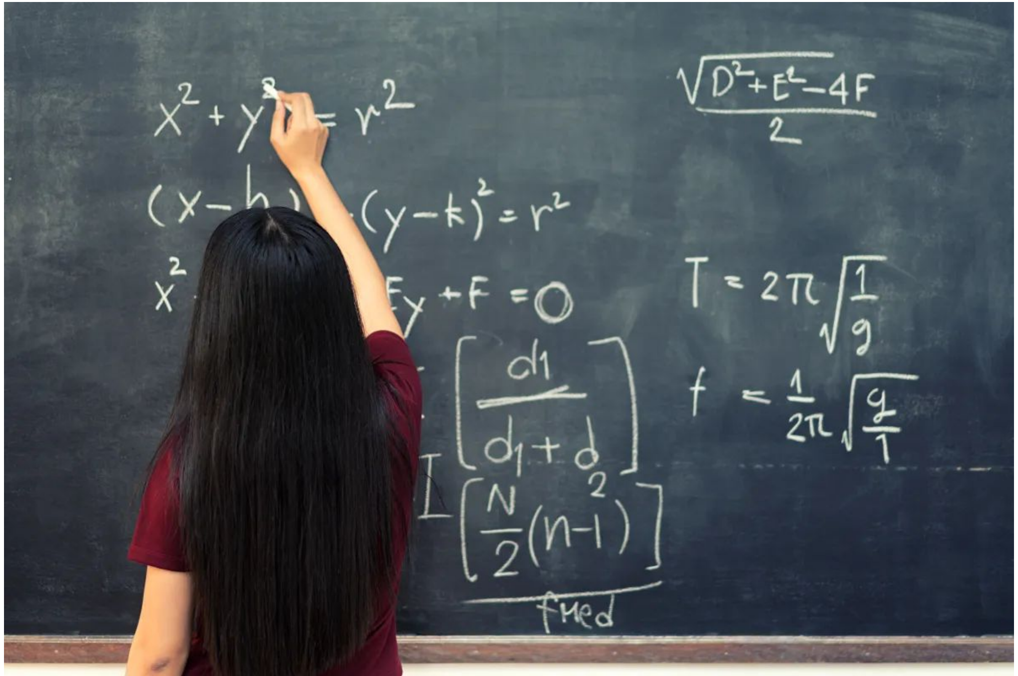
Where Math Comes From (And Why It Matters For Your Teaching)

Brett Fawcett

How many Math teachers are here?

What about non-Math teachers?

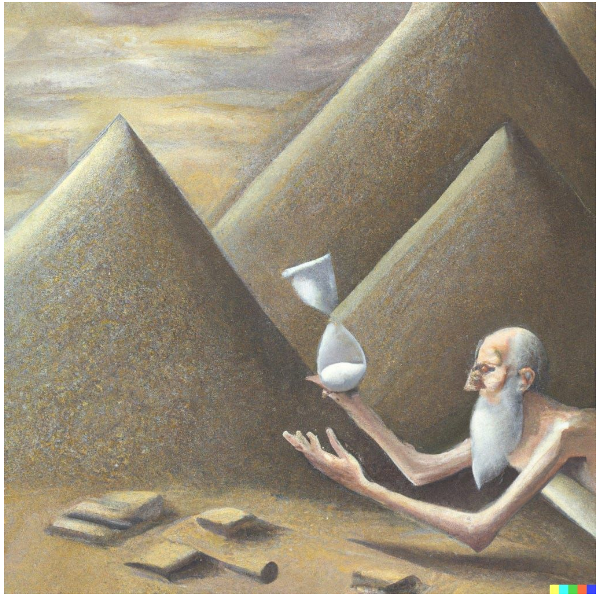


Who I Am

I teach Philosophy, History/Social Studies, English Language Arts/Literature, Religion/Theology, Latin, and Physics at **St. Isidore Learning Centre** and Chesterton Academy. (My wife is an early childhood educator.)

So I'm not a math teacher *per se*, though I covered a Math teacher's class for a week and was getting ready to take it over if I had to.

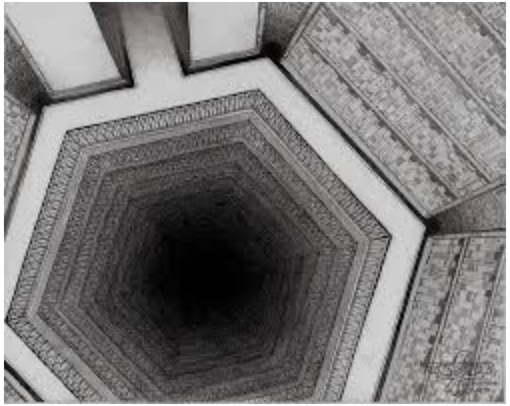
(We talked about polynomials and how many grains of sand it would take to fill the universe. Archimedes helps us understand William Blake's poetry.)

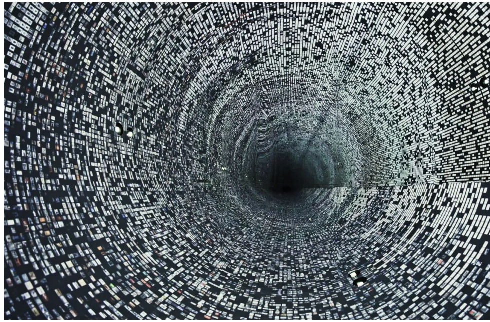


I’m not a Math teacher *per se*,

So this is a non-math teacher’s attempt to show how Math might be made interesting to all students (and all teachers).

And I’ll try to show you some cross curricular connections along the way. (For example, give your students some of Jorge Luis Borges’ stories to prepare them to think about infinity.)





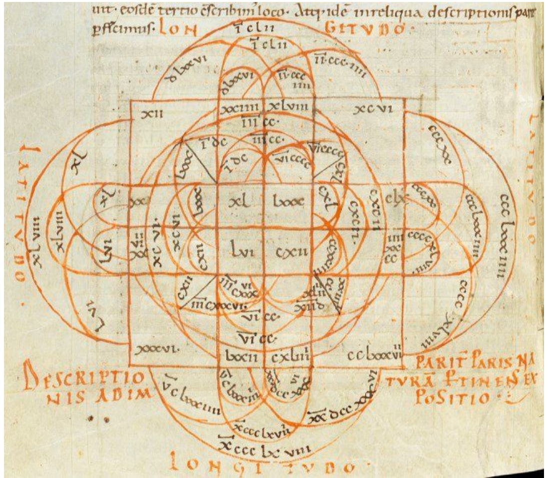
Why should students learn math?

Is math more like art or like science?

In the ancient and medieval curriculum, math was considered a **liberal art**.

A “science” was a body of knowledge, but an art was a skill you had, and liberal arts were the skills of someone who was “free” (i.e. not a slave).

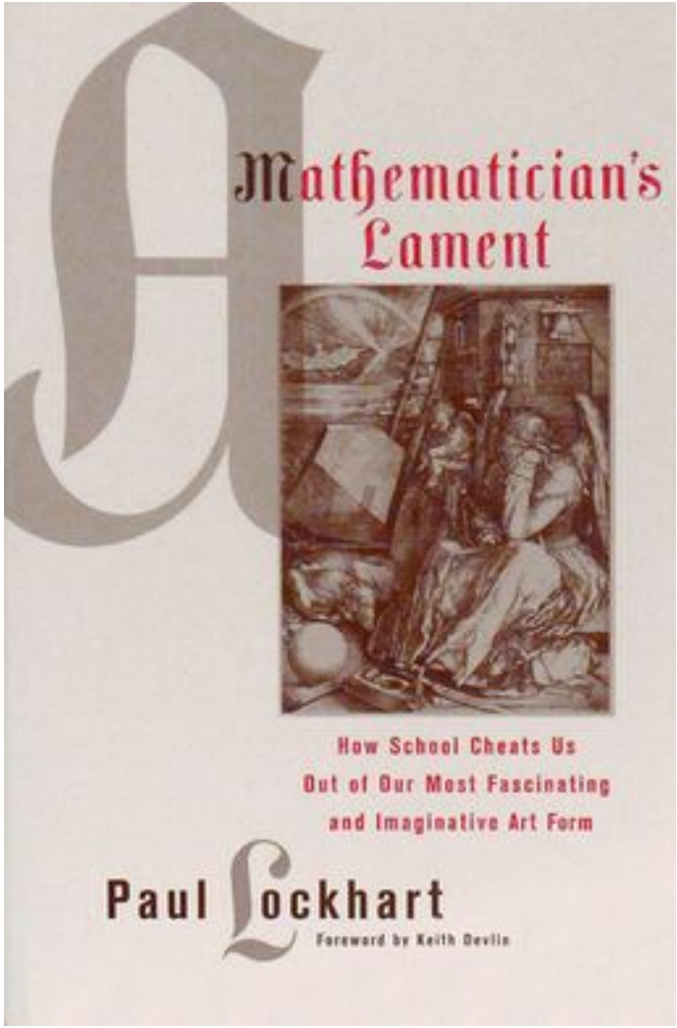
Why do you think learning math was considered **liberating**?



Paul Lockhart says:

If we're just training students to correctly follow rules without understanding them, we're training industrial slaves who know how to follow orders uncomprehendingly, not producing free, rational people.

(Lockhart holds a Ph.D in mathematics and has taught math at both the university and the pre-K-12 levels.)



Imagine everyone has to learn music starting in elementary, but this doesn't mean actually listening to music: it means learning the notation. Students describe it this way:

"Music class is where we take out our staff paper, our teacher puts some notes on the board, and we copy them or transpose them into a different key. We have to make sure to get the clefs and key signatures right, and our teacher is very picky about making sure we fill in our quarter-notes completely. One time we had a chromatic scale problem and I did it right, but the teacher gave me no credit because I had the stems pointing the wrong way."

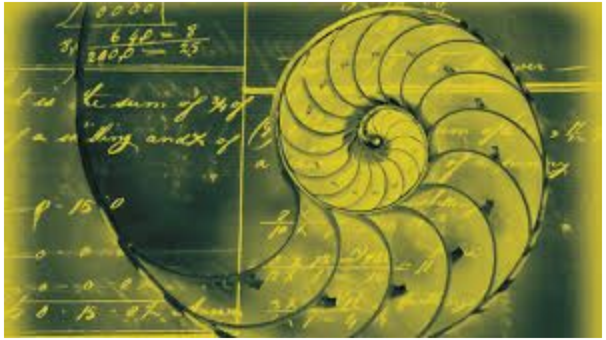
Students must take courses in Scales and Modes, Meter, Harmony, and Counterpoint. People reason: *"It's a lot for them to learn, but later in college when they finally get to hear all this stuff, they'll really appreciate all the work they did in high school."*

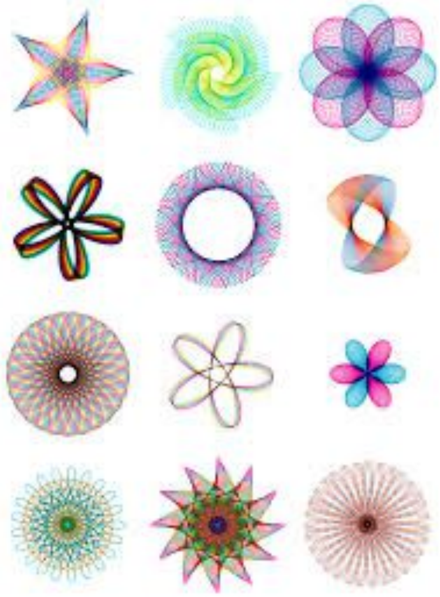
For Lockhart, this is too often how we teach mathematics. But what we miss is what the medievals understood: **Mathematics is an art**.

In fact, as G.H. Hardy put it in *A Mathematician's Apology*, "Real mathematics must be justified as art if it can be justified at all."

And, like all art, the criteria is **beauty**. As Bertrand Russell put it, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty."

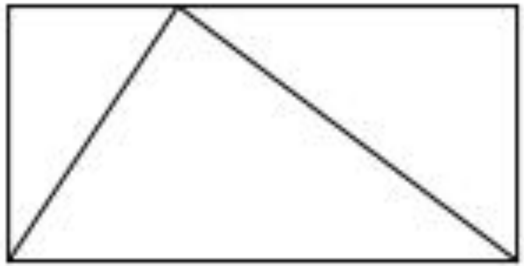
Do you think math is beautiful? You need to see that beauty before your students can.



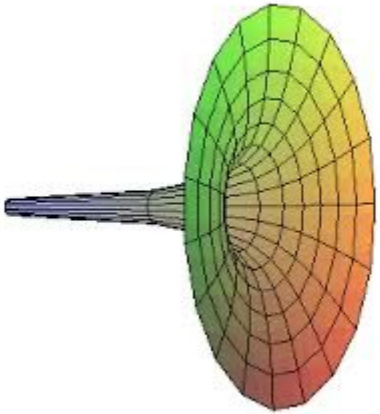


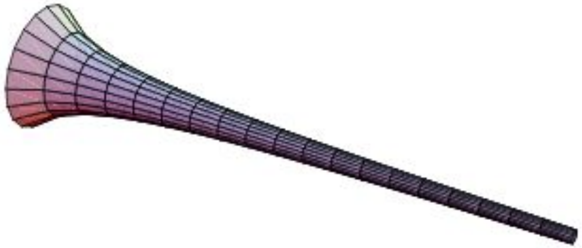
"A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*." (Hardy's *Apology*)

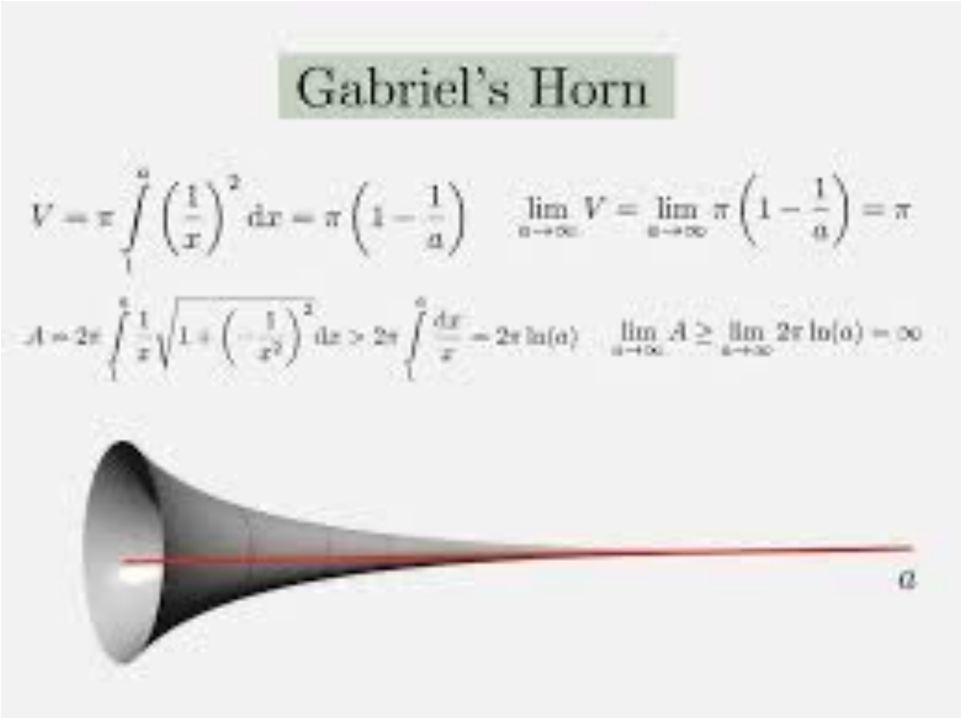
Here's Lockhart's example: I can imagine a triangle in a box. "I wonder how much of the box the triangle takes up? Two-thirds maybe? The important thing to understand is that I'm not talking about this drawing of a triangle in a box…There's no ulterior practical purpose here. I'm just playing. That's what math is— wondering, playing, amusing yourself with your imagination. For one thing, the question of how much of the box the triangle takes up doesn't even make any sense for real, physical objects…The mathematical question is about an imaginary triangle inside an imaginary box. The edges are perfect because I want them to be— that is the sort of object I prefer to think about. This is a major theme in mathematics: things are what you want them to be. You have endless choices; there is no reality to get in your way."



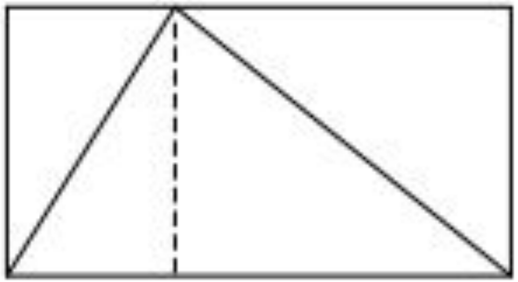
For example: Gabriel's horn, a geometric figure that has infinite surface area but finite volume.







"So we get to play and imagine whatever we want and make patterns and ask questions about them. But how do we answer these questions? It's not at all like science. There's no experiment I can do with test tubes and equipment and whatnot that will tell me the truth about a figment of my imagination. The only way to get at the truth about our imaginations is to use our imaginations, and that is hard work.



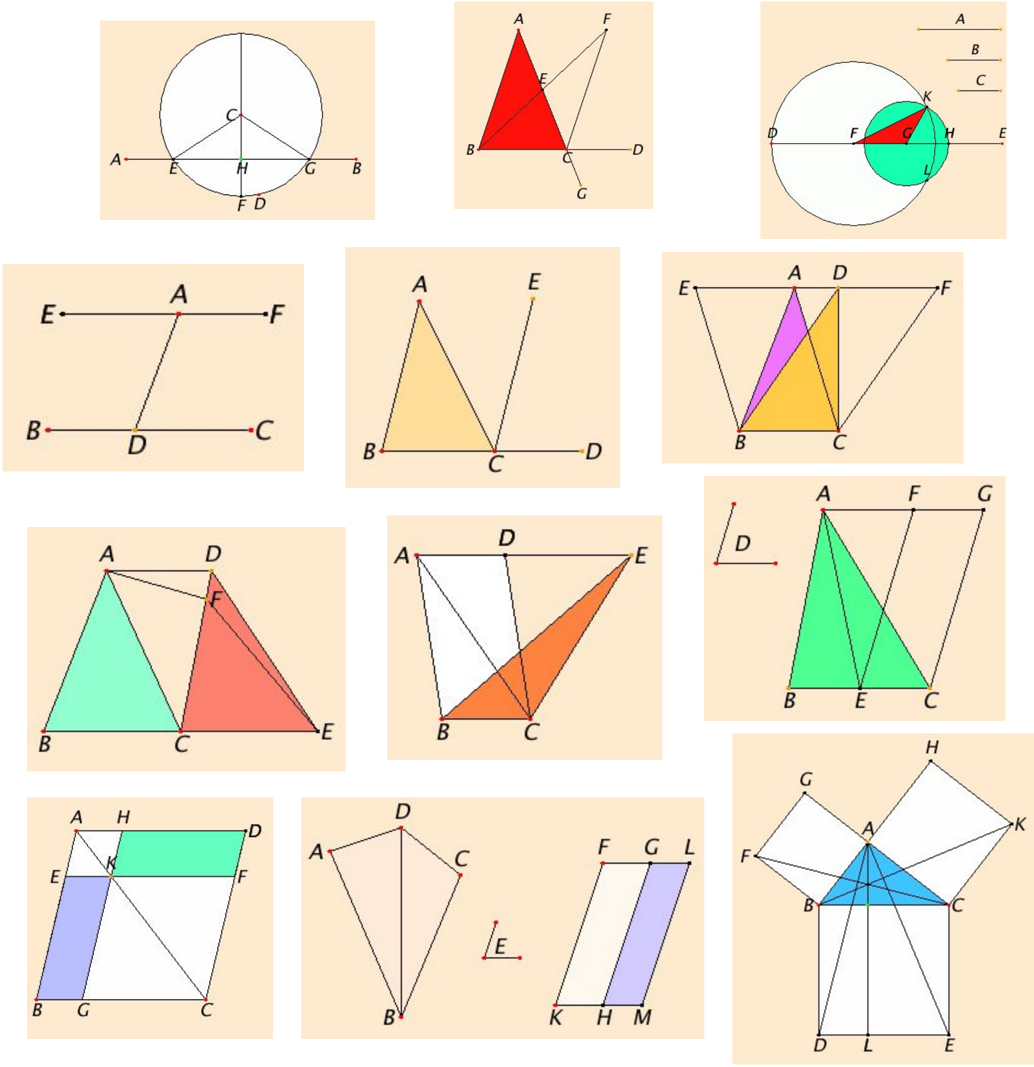
In the case of the triangle in its box, I do see something simple and pretty: If I chop the rectangle into two pieces like this, I can see that each piece is cut diagonally in half by the sides of the triangle. So there is just as much space inside the triangle as outside. That means that the triangle must take up exactly half the box!

This is what a piece of mathematics looks and feels like. That little narrative is an example of the mathematician's art: asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations. There is really nothing else quite like this realm of pure idea; it's fascinating, it's fun, and it's **free**!"

This actually used to be how geometry was taught in the classical and medieval world.

Euclid's *Elements* (written 300 B.C. and still used as a textbook today) starts with a handful of axioms, and then proceeds to draw implications from them for 7 books. He doesn't just use the Pythagorean theorem; in 47 propositions, he **proves** it by playing with shapes based on the principles he's established. (He does this with quadratics, too.)

You have to follow along as he builds up a logical chain of argument, but it's very satisfying when you see it all come together. You don't just "know" or "use" the theorem, but you understand it in a deep and meaningful way. It trains the mind in reasoning, not just in using tools.



Euclid does quadratics in the second book of the Elements using geometry.

Proposition 1.
If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

Proposition 2.
If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.

Proposition 3.
If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.

Proposition 4.
If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

Proposition 5.
If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.

Proposition 6.
If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Proposition 7.
If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.

Proposition 8.
If a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line.

Proposition 9.
If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.

Proposition 10.
If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.

Proposition 11.
To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

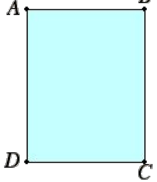
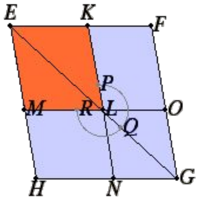
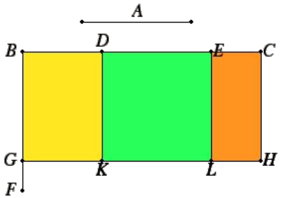
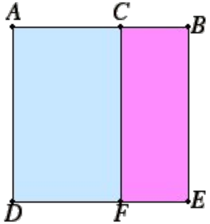
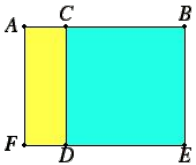
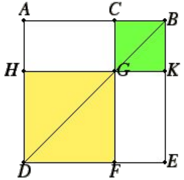
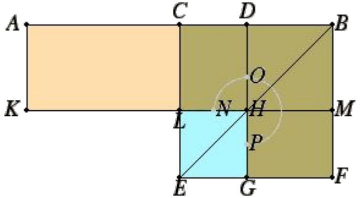
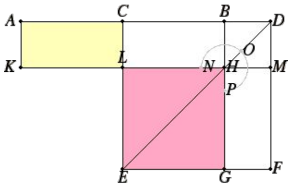
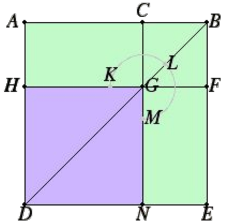
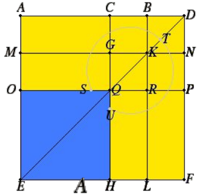
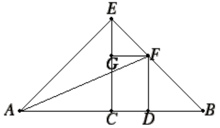
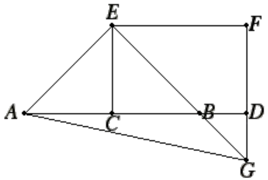
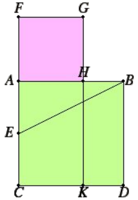
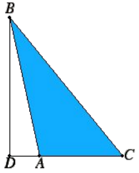
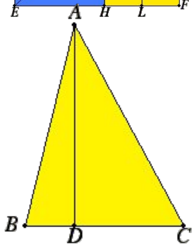
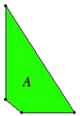
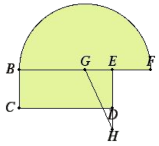
Proposition 12.
In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Proposition 13.
In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Proposition 14.
To construct a square equal to a given rectilinear figure.

Definition 1.
Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle.

Definition 2
And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon.*

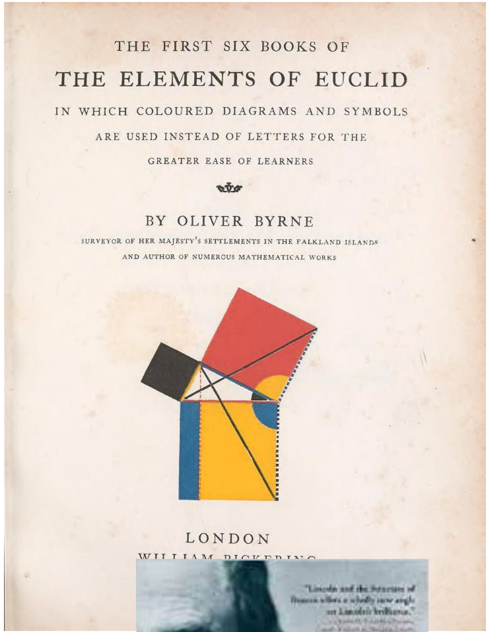


















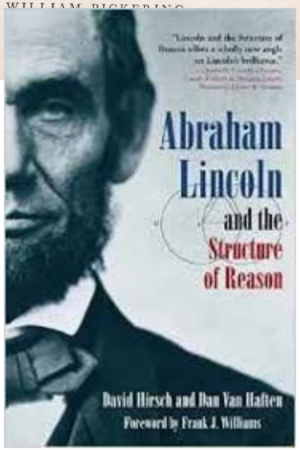
*"Euclid alone has looked on **Beauty** bare.*
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere In shapes of shifting lineage…"
(Edna St. Vincent Millay - an early feminist poet)

When Abraham Lincoln decided he wanted to be a lawyer, and recognized the shortcomings of his own education, he went and studied Euclid in detail before going into legal studies. He said it was so he could learn how to demonstrate things, as shown in the *Lincoln* movie:

"Euclid's first common notion is this: 'Things which are equal to the same thing are equal to each other.'... Euclid says this is 'self-evident.' D'you see? There it is, even in that two-thousand year old book of mechanical law: it is a self-evident truth that things which are equal to the same thing are equal to each other. We begin with equality. That's the origin, isn't it? That balance, that's fairness, that's justice."

Do you see how **humanizing** and **liberating** the beauty of geometry is, especially if you teach it like Euclid did?

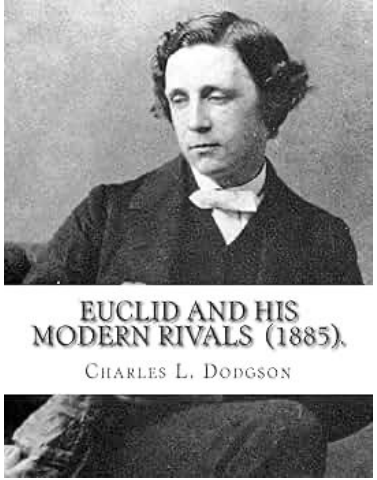


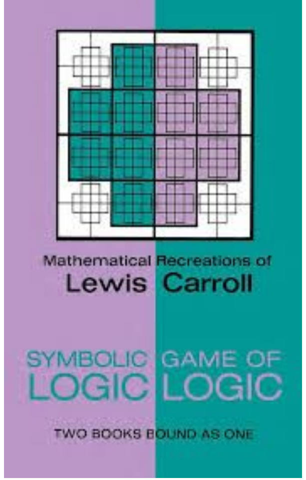


Lewis Carroll, the Anglican deacon who wrote *Alice in Wonderland* and *Through the Looking Glass*, was also a logician and mathematician; Wonderland can be seen as depicting the horrors of a world without logic, which shows the importance of mathematical reasoning.

He also wrote a play criticizing modern geometry textbooks and calling for a return to the *Elements* called *Euclid and His Modern Rivals*.

Euclid's world is the opposite Wonderland.





Takeaway:

If students just know how to follow orders, we're training slaves.

If students understand the meaning and logic behind the math they're doing, we're producing rational, free artists.

This is why mathematics used to be considered a **liberal art**.

Socrates and a slave boy doing geometry

Servile arts (or **mechanical arts**): These were skills necessary for sustaining life and getting a job (i.e., service): Weaving, navigation, blacksmithing, agriculture, hunting, medicine, cooking, etc.

Liberal arts: These were the skills of a free (*liber*) person; they had inherent valuable and were self-justifying

If the reasoning behind learning math is that “it helps you get jobs,” that turns math into a servile art. But the classical/ancient/medieval perspective was that it was a liberal art.

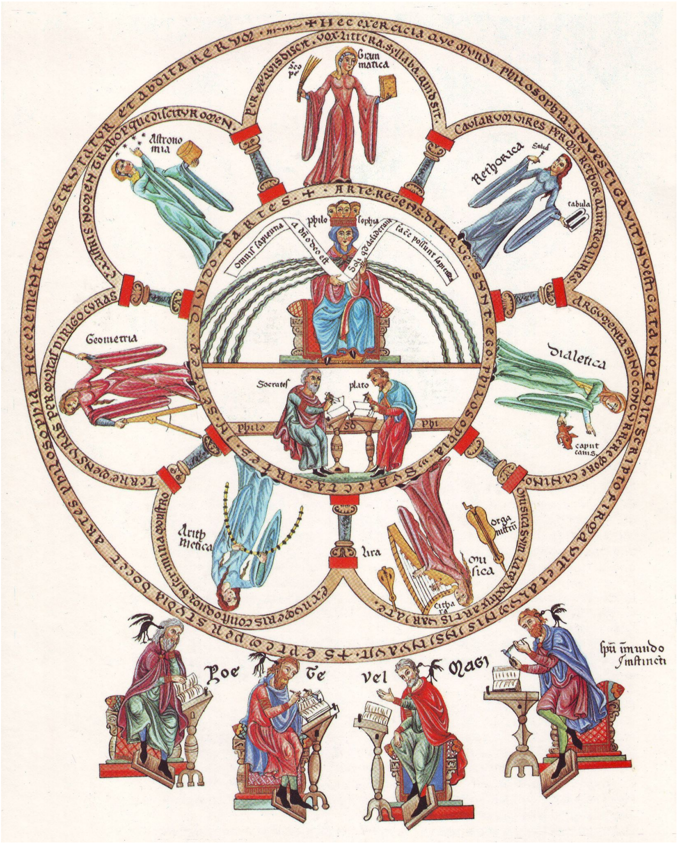
In the Middle Ages:

The first three liberal arts (the trivium) were grammar, logic, and rhetoric.

The next four (the quadrivium) were **arithmetic**, **geometry**, **music**, and **astronomy**. (Euclid was the main text for learning geometry.)

Every educated person had to learn these before they could study theology, philosophy, medicine, or law.

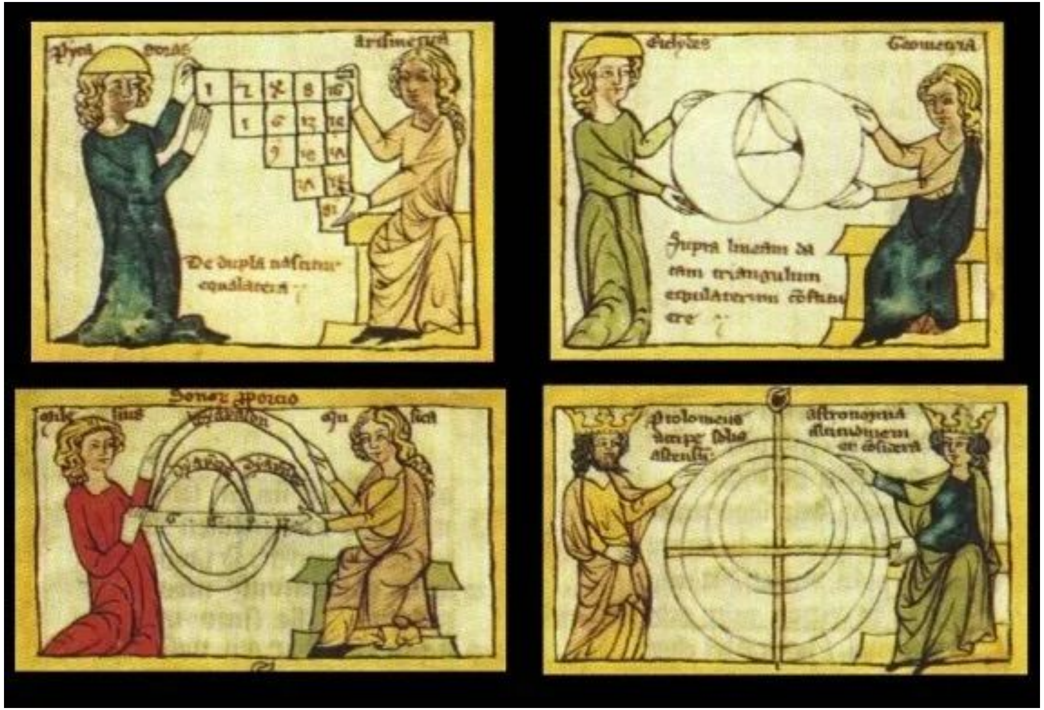
Why do you think these four were considered so fundamental?



Arithmetic: Unmoving, eternal numbers
Music: Numbers in motion

Geometry: Eternal, unmoving shapes
Astronomy: Shapes in motion

To understand the medieval worldview that gave rise to this form of pedagogy, we need to go a little bit deeper–which will take us into modern philosophical foundations of mathematics, which may be deeper than you bargained for, but which shows us *why* math is so fundamentally beautiful.



WARNING

Thinking about this correlates to people dying suspiciously and possibly being murdered by an esoteric cult, losing their sanity and ending up in sanitariums or starving to death, and collapsing into despair and hopelessness as their life's work crumbles before their very eyes.

So if you want to go get a muffin now, I'll understand.

Are numbers real?

Do they exist outside of our minds, or did we just make them up as a tool for measuring stuff?

This matters because think about how we teach literature differently than we teach about the physical sciences. There’s a difference between learning grammar (which we more or less made up) and kinematics (which exists outside of our minds).

It seems like, even if there were no material universe, 2+2 would still =4.

That feels like an eternal, necessary, timeless truth.

But that would mean numbers are eternal, immaterial things that have to exist in some Platonic realm somewhere, and that seems weird, too. How would they interact with our physical world?

Here’s how philosophers deal with it.

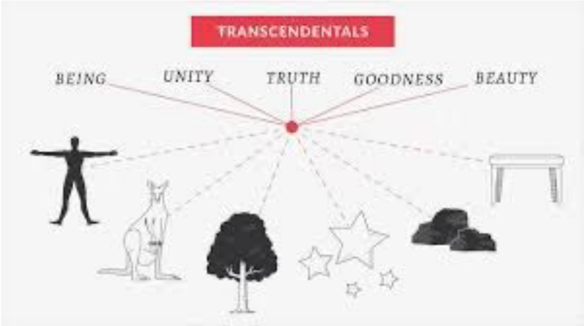
The Transcendentals

There are lots of beings out there, but what do all beings have in common?

The transcendentals are what the philosophers identified as those properties which are common to all being.

These are usually identified as truth, goodness, beauty, and **<u>unity</u>**.
Let's hone in on unity, or oneness.
Think about this:

Everything that exists is **a** thing. It is (whatever else it is) **one** thing, whether that's a person, a crowd, a pencil, a bird, a country, an atom–all of these have **unity** that make them <u>a</u> thing, <u>an</u> example of whatever they are.



Thus the ancient and medieval definition of “number” (which is what Euclid accepted) was:

A collection of units.

Zero and one were not numbers; the smallest number was two, which is a collection of units.

Numbers proceeded from unity, and every number also was a unit. Solving math problems meant reducing them to unity.

“Computation is that action of the mind whereby things are referred to unity” (Kirkby’s *Arithmetical Institutions*, 1735)

The idea was that **number proceeds from unity and returns to unity**.

Euclid's Elements

Book VII

Definition 1

A unit is that by virtue of which each of the things that exist is called one.

Definition 2

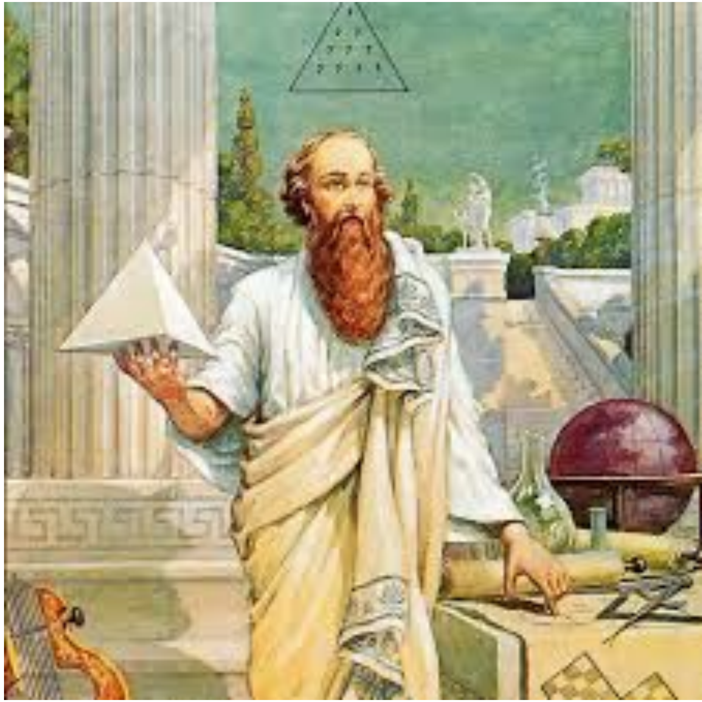
A number is a multitude composed of units.

Pythagoras

–Sometimes regarded as the first pure mathematician, and supposedly coined the word “philosophy”

–Founded a school of mathematical research that was also a cult; because they didn’t believe in writing down their secret teachings, they are fairly mysterious, but it appears they believed in reincarnation and practiced vegetarianism (although eating beans was forbidden), and lived a communal lifestyle

–Hard to know what’s legendary and what’s biographical about him



Pythagoras believed the deepest reality was “the One,” the Monad, the Unit, and that everything else proceeded from the One (think about how unity must be the primordial transcendental). Basically, the Monad was God and God was the Monad.

Supposedly he taught:

“That <u>the monad was the beginning of everything</u>. From the monad proceeds an indefinite duad, which is subordinate to the monad as to its cause. That <u>from the monad and the indefinite duad proceed numbers</u>. And from numbers signs. And <u>from these last, lines of which plane figures consist. And from plane figures are derived solid bodies</u>. And from solid bodies sensible bodies, of which last there are four elements; fire, water, earth, and air. And that the world, which is endued with life, and intellect, and which is of a spherical figure, having the earth, which is also spherical, and inhabited all over in its centre, results from a combination of these elements.”

This sounds mystical, but think about it–let’s work backwards:

The earth is a sphere, which is a shape. You can imagine shapes without planets, but you can’t imagine planets without shapes, so geometry comes before astronomy.
A sphere is a shape. You can have numbers without shapes, but you can’t have shapes without numbers (a triangle has 3 lines, so it depends on the number 3; but you can imagine 3 without triangles). So geometry depends on arithmetic.

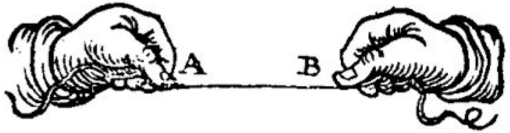
Also, a shape depends on lines. As Euclid said, a line is the shortest distance between two points. A point is that which has no parts–it is a simple unit. From the unit (the point) proceed lines, from lines proceed shapes, from shapes proceed objects, etc. etc. Unity is again original.

Thus, Pythagoras said, numbers are the ultimate reality that govern the universe. This view is also called Platonism.

Definition 1.
A *point* is that which has no part.

Definition 2.
A *line* is breadthless length.

Definition 3.
The ends of a line are points.



He had further proof of this.

Supposedly, Pythagoras once walked past a forge and heard two different sizes of hammers producing different sounds that harmonized with each other. He found that the tune of hammer is directly proportional to the size of the respective hammer, which makes the two hammers’ frequency one doubled another.

He therefore developed a musical scale based on **ratios**. His insight was that music was fundamentally mathematical.

He further suggested that the distance between the planets and their motion were in proportions that matched his scale, and thus came up with the idea of “the music of the spheres.” Once again, number ruled the universe.

Arithmetic and geometry led to music which led to astronomy. **See how all of this influenced the quadrivium?**

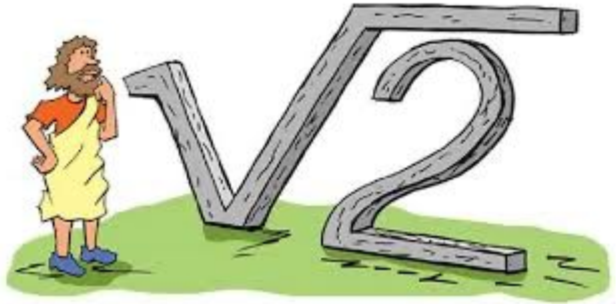
The problem:

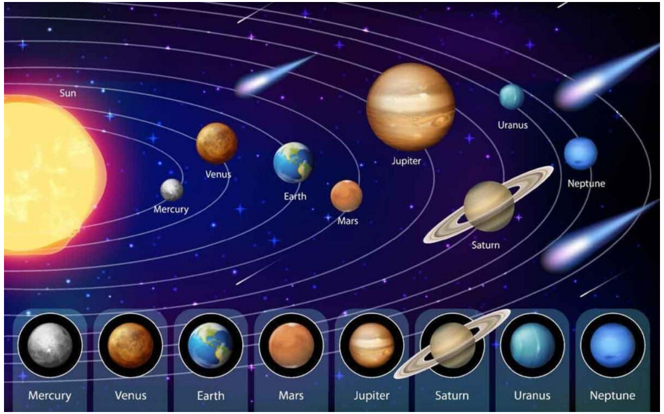
This relied on all numbers being rational.

A Pythagorean named Hippasus discovered that $\sqrt{2}$ had to be irrational and thus that irrational numbers existed. This apparently horrified the Pythagoreans, and it didn't help that he shared his discovery publicly.

He later drowned, supposedly as a punishment from the gods–though some suspected it was a punishment from his fellow Pythagoreans.

We also know now that the planets do *not* move in time with a musical scale.

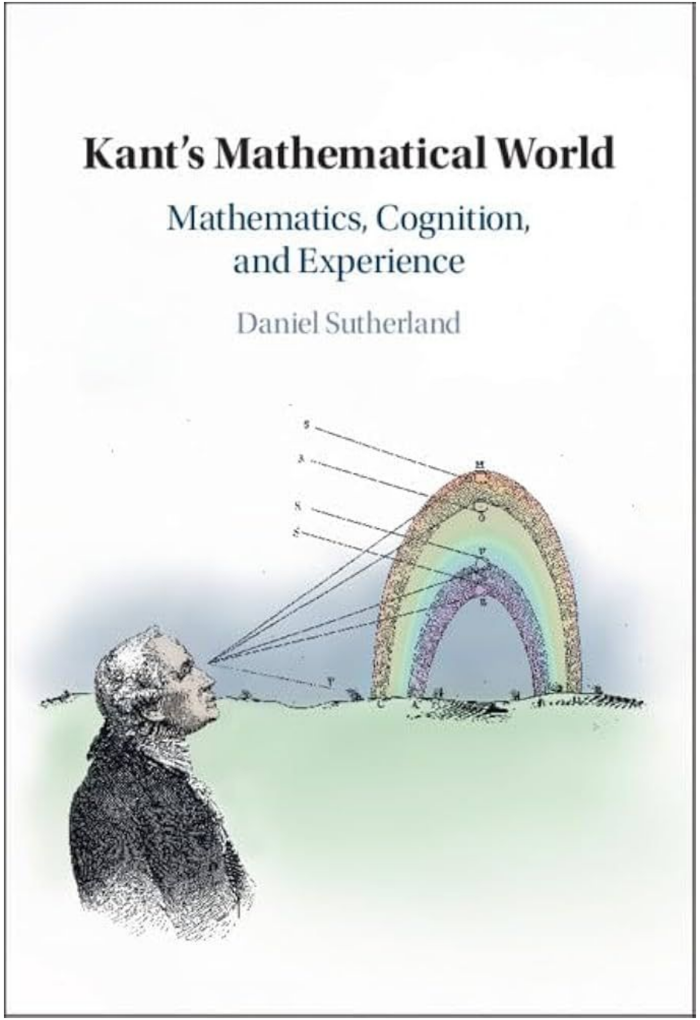




There was a movement in philosophy and mathematics to say that this old “realist” or “Platonist” view of numbers held by Pythagoras and his heirs doesn’t work.

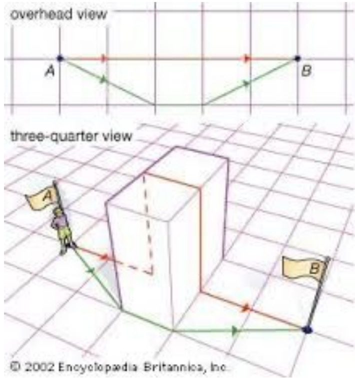
“Intuitionism” says that math is just the way our minds think and has no necessary connection to the real world (Immanuel Kant, E.J. Brouwer). If you can’t construct a mathematical proof for something, it doesn’t exist.

“Logicism” or “formalism” is the idea that math is basically just applied logic, and it has rules, but it isn’t like numbers themselves have a real existence.



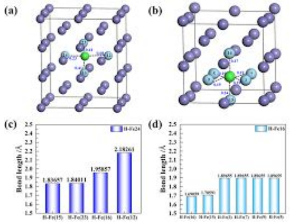
HOWEVER:

In the 20th century, the physicist Eugene Wigner wrote a seminal essay called “The Unreasonable Effectiveness of Mathematics in the Physical Sciences.” It turns out that physical science tracks really, really well with math, which is weird if we just made up math.



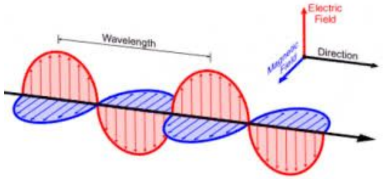
Just a few examples:

–Born and Heisenberg played around with matrices (rectangular arrays of numbers or other mathematical objects) in an abstract way, but these turned out to perfectly describe the hydrogen atom



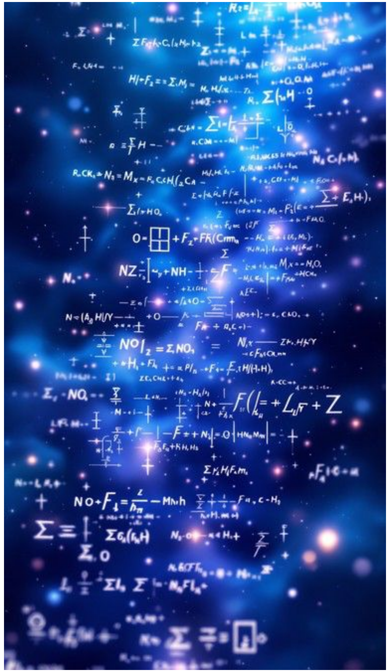
–Maxwell’s equations turned out to perfectly describe radio waves, which were only discovered at the time of Maxwell’s death

–Non-Euclidean geometry was developed as a mental experiment but turned out to be extremely useful for Einsteinian physics



Wigner was not religious, but called this correspondence between math and physical science a "**miracle**" in need of an explanation.

Paul Dirac, a mathematician and physicist who hated religion and claimed not to believe in God, also couldn't talk about this without using religious language, writing:



"It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it. You may wonder: Why is nature constructed along these lines? One can only answer that our present knowledge seems to show that nature is so constructed. We simply have to accept it. One could perhaps describe the situation by saying that God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe."

He also said that a mathematical equation is more likely to work if the mathematician is trying to make it as beautiful as possible.

I can only dip into this, but here’s something you may want to delve more into:

Modern mathematics is founded on **set theory**, which was originally developed by Georg Cantor. This is the idea that every mathematical object is some kind of a set, which Cantor called “**a many that allows itself to be thought of as a one” (not unlike the classical definition of a number)**. A geometrical object, for example, is an infinite set of points.

Cantor showed that infinities can be different sizes. After all, the amount of even numbers is infinite, but the total number of integers is also infinite, and that’s a bigger infinity than the totality of even numbers; he also showed that the infinity of real numbers is larger than the infinity of rational numbers.

$$\aleph_0$$

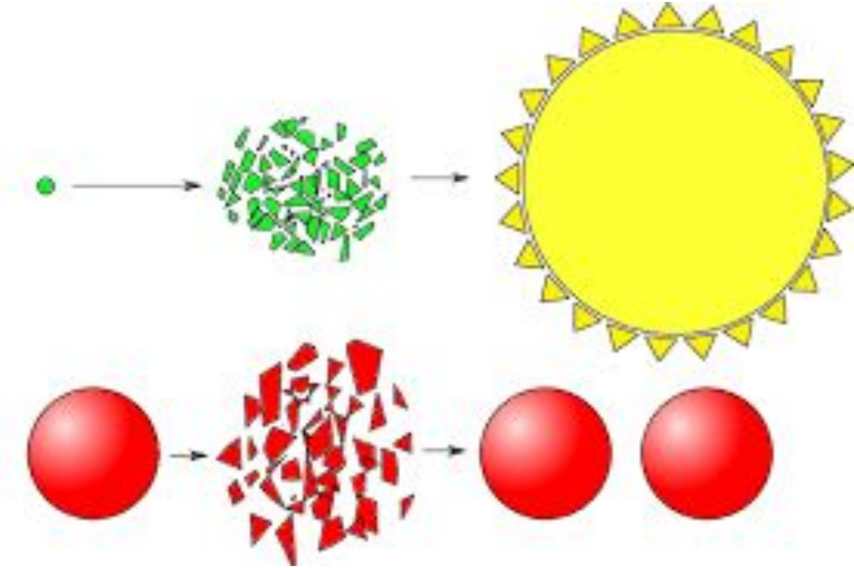
The symbol used by Cantor to denote the smallest infinity, equal to the size of the set of counting numbers

A wild consequence of set theory:

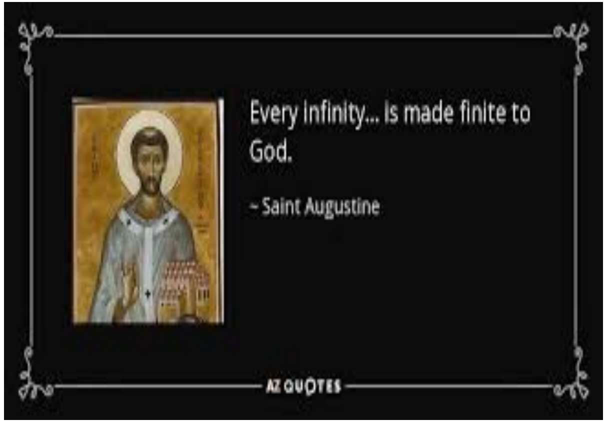
The Tanach-Barski paradox:

Theoretically, you could disassemble a bowling ball into subsets of points, assemble a new bowling ball out of some of those points, and then reassemble the leftover points back into another bowling ball that was identical to the original - a paradox of infinity.

(Religion teachers: Is this what Jesus did with the loaves and fishes?)

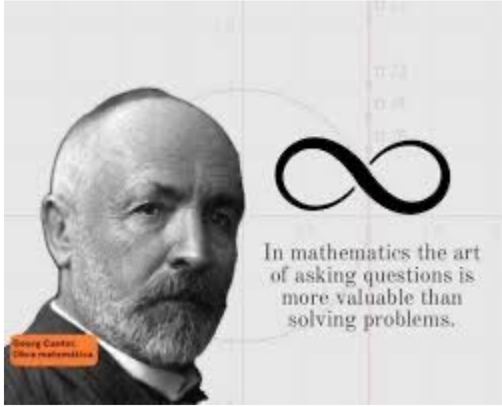


This was controversial, because, since Aristotle, mathematicians had rejected the idea that infinity was a useful or possible concept. However, Christian philosophers like Augustine *had* been willing to deal with infinity (i.e. Augustine says God knows an infinite number of things).



Indeed, Cantor said that, behind the "infinity of infinities" he had discovered, there had to be a proper "absolute infinity" that grounded all of them, and he was clear that this was God, Who he thought had revealed set theory to him (he corresponded with the Vatican about this). This is similar to Pythagoras' idea of the Monad.

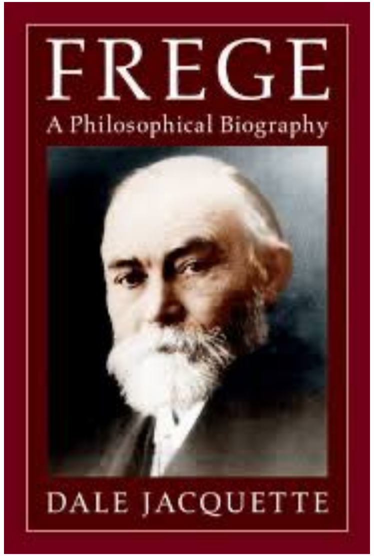
Cantor was initially rejected by mathematicians which drove him to despair and mental breakdown and led him to a sanitorium. Nowadays, as David Hilbert said, mathematicians accept set theory and "no one will cast us out of Cantor's paradise."

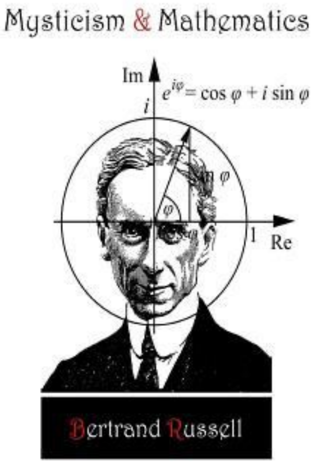


20th century mathematicians tried to build up a purely logical mathematics based on set theory. Frege, the founder of analytic philosopher, tried this, but Bertrand Russell found a flaw in his reasoning known as "Russell's paradox."

Russell sent it to Frege right when Frege was about to publish his book on mathematical foundations. Frege couldn't answer it and published the book with a postscript admitting that Russell had basically refuted the whole thing already.

Russell (and Whitehead) then wrote a three volume work on the foundations of mathematics where they tried to do better than Frege, where they took 372 pag to prove that 1+1=2, a "proposition [which] is occasionally useful."





*54·43. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51·231] $\equiv . \iota' x \cap \iota' y = \Lambda .$

[*13·12] $\equiv . \alpha \cap \beta = \Lambda$ (1)

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda$ (2)

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Then along came Kurt Gödel.

Gödel was really interesting.

- He is often considered the greatest logician of the 20th century.
- After a conversation with Einstein, he showed how general relativity meant that time travel to the past was possible (which made Einstein doubt his own theory).
- He told an American immigration judge he had found a flaw in the American constitution that would allow fascism.
- He developed an argument for God’s existence which has been proven to work by computers.
- He was extremely reclusive and paranoid and would only eat food cooked by his wife so he wouldn’t he poisoned, so when she went to the hospital he starved to death.

Gödel also exploded Russell's project the way Russell exploded Frege's.

Basically, he made a mathematical version of this sentence:

"This statement is unprovable."

- If it's provable, it's false, which means mathematics is inconsistent and illogical.
- If it's true, it can't be proven, which means there are true statements in mathematics that can't be proven.

(For example: the continuum hypothesis, which cannot be proven or disproven by the axioms of arithmetic.)

So math can't be a purely logical enterprise. Gödel believed numbers had a real, angelic kind of extra-mental existence *a la* the Pythagoreans and Platonists.

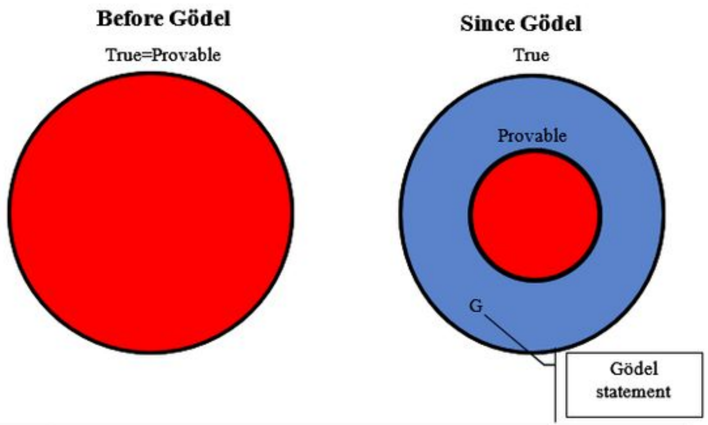
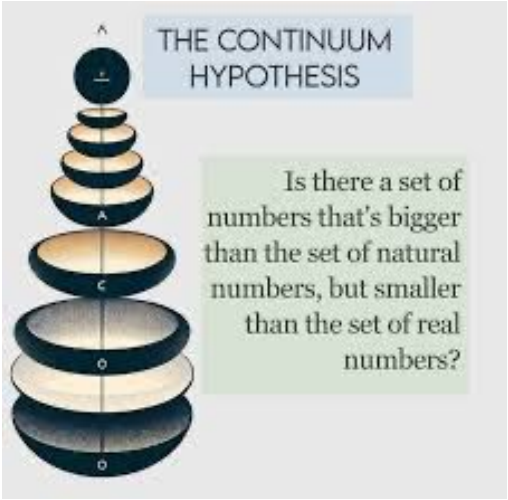


Figure 1. Before Gödel, all true statements in basic arithmetic were thought to be provable. Gödel showed that this view was wrong and that there are statements that are true but not provable.



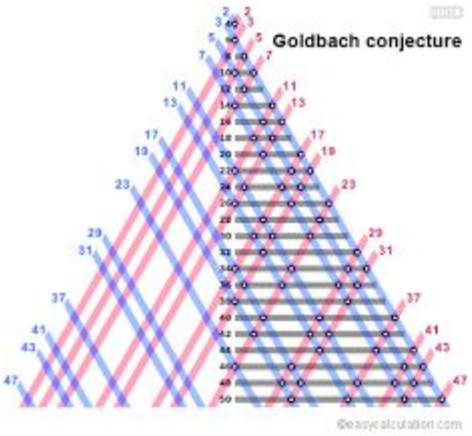
So what does this mean?

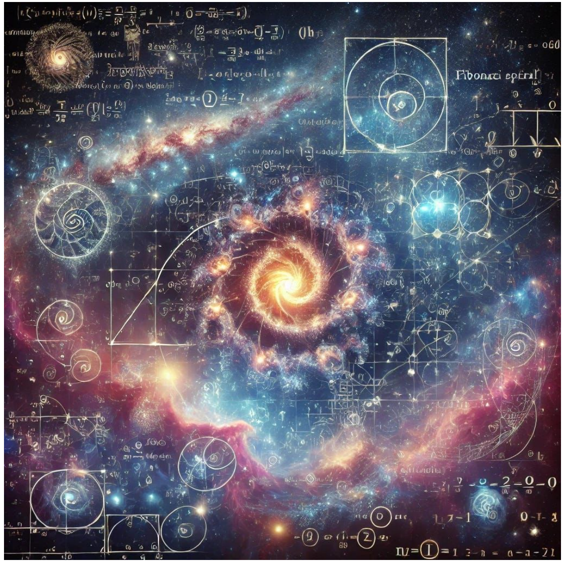
How do we teach physical science?
Theoretically, it’s “useful” for some students’ careers.

But fundamentally we teach it in a way that shows that the universe is a beautiful, interesting thing external to ourselves that we enter into and that we want to understand so we can appreciate it and play around in it. **It pulls us out of ourselves.**

This was the ancient and medieval viewpoint of math, and modern mathematics and philosophy seems to confirm it.

So perhaps we should also teach math as if it’s a world of mysterious but beautiful realities bigger than our minds that we get to play around in.





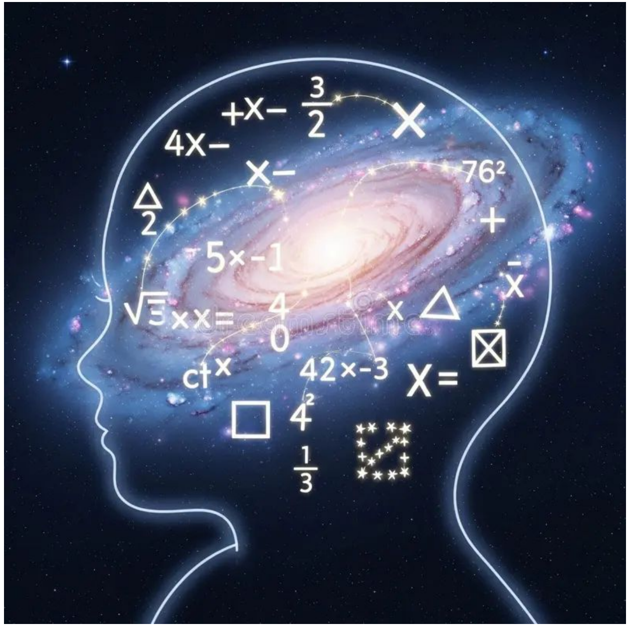
In summary:

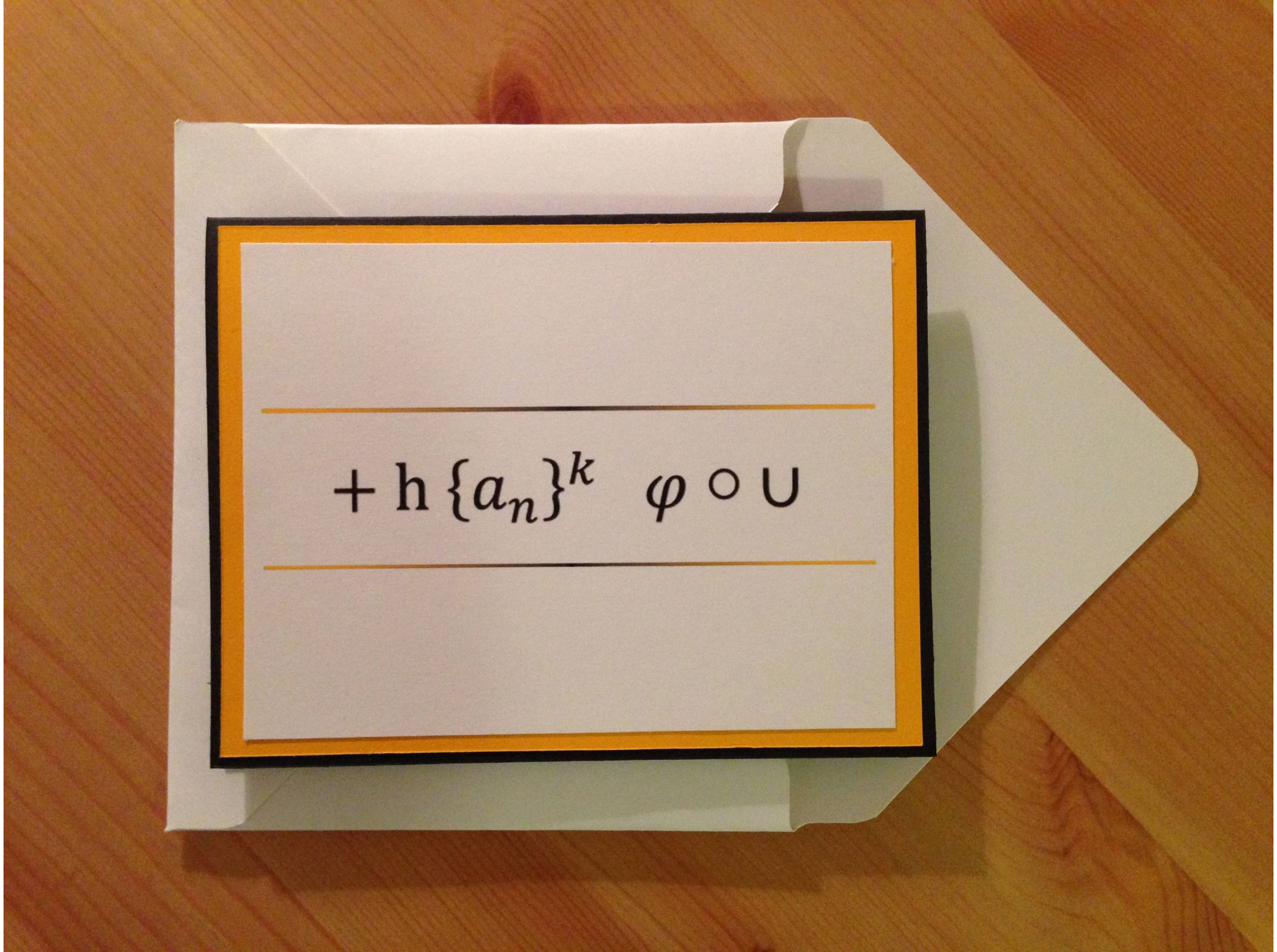
–If students are just following mathematical rules without understanding them, they're essentially doing math like slaves.

–Allowing students to see the meaning and reasoning behind mathematical formulae is liberating; students do math like free people, not like slaves.

–Comprehending the deep meaning behind mathematics shows the **beauty** of mathematics, which is also liberating: beauty is desired for its own sake, not as a tool to get a job.

–Mathematics is a mysterious, mystical world that's bigger than our minds, and it should feel that way in our classrooms.




$+ \mathrm{h} \{a_n\}^k \quad \varphi \circ \cup$

BONUS MATERIAL

Brett Fawcett

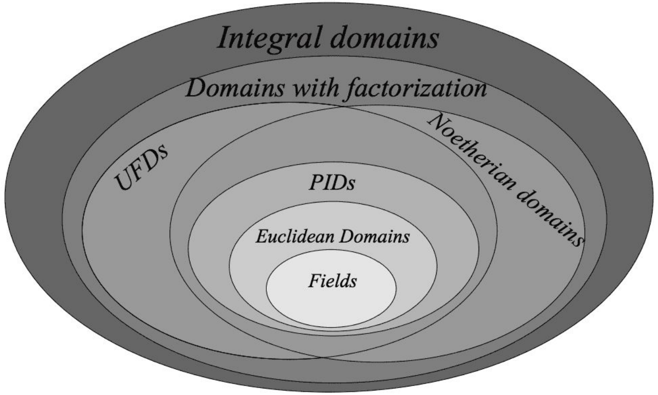
Many mathematicians thought that you could only prove something in math by constructing it mathematically, since math is just a human product.

When David Hilbert solved a major regarding polynomial rings and ideals with is basis theorem, he used non-constructive methods. Paul Gordan initially rejected this, saying:

“This isn’t mathematics. This is theology.”

(I.e. this is treating math as something other than the product of our mental processes–something non-material that exists outside our minds)

Later, when Hilbert’s solution was accepted to work, Gordan stated, “After all, I have convinced myself that **even theology has its merits.”**



Frege's idea is that a set could be any collection of anything.

Russell challenges this:
Can you have a set of all sets which are not members of itself?

Analogy:
The barber shaves everyone who doesn't shave himself.
Does the barber shave himself?
If he doesn't shave himself, then he's shaved by the barber, which means he DOES shave himself.

Similarly:
If a set contains all sets which are not members of itself:
–If it is not a member of itself, then that means it must be a member of itself–CONTRADICTION
–If it is a member of itself, then it cannot be a member of itself–CONTRADICTION

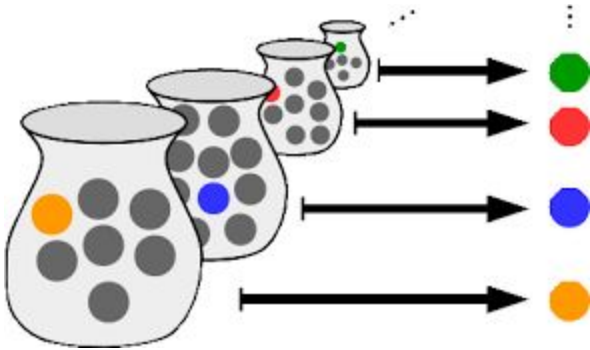
What this proves is that there has to be a category above sets, a "set of all sets" or class. But if all mathematical objects are sets, then whatever category is above sets must not be a mathematical object–which means mathematics has to be based on philosophy (or is it theology?).

In the Middle Ages, St. Thomas Aquinas had already recognized that you could have infinities of different sizes (sort of).

He said that **angels were "relatively infinite"** the same way that colour was relatively infinite, i.e., they are finite in their being (that is to say, compared to the infinity of God) but infinite in the sense that they are not limited to a particular finite material thing. He compared this to colours: "Whiteness" is infinite in the sense that it is not limited in and of itself (though it is instantiated in particular finite white things).

Thus, **God is absolutely infinite (or "simply infinite" as Aquinas puts it) but angels are relatively infinite**. This was centuries before Cantor proved there were different sizes of infinitities.

The Axiom of Choice



“Given a collection of (non-empty) sets, we can choose one element from each set.”

This seems reasonable, but what if it’s an infinite collection of non-empty sets? This is why the axiom of choice has been challenged: a human mind couldn’t possibly construct this, so from some perspectives it isn’t mathematically valid. [***An infinite mind could do it, of course…***]
Do you get why this leads to the Tanach-Barski paradox?

Yet Zorn’s lemma (every non-empty partially ordered set in which every totally ordered subset has an upper bound contains at least one maximal element) is the basis of a bunch of important mathematical work (e.g. Hilbert’s Nullstellensatz which links geometry and algebra).

There’s an hour-long art film about Zorn’s lemma, if that’s your jam.

Gödel’s ontological argument for God’s existence:

- **Axiom 1:** If φ is a positive property, and if it is necessarily true (true in all possible worlds) that every object with property φ also has property ψ, then ψ is also a positive property.
- **Axiom 2:** The negation of a property φ is positive if, and only if, φ is not positive.
- **Theorem 1:** If a property φ is positive, then it is possible that there exists an object x that has this property (in at least one possible world, there exists an object x that has this property).
- **Definition 1:** An object x is God-like if, and only if, x has all positive properties.
- **Axiom 3:** The property of being God-like is itself a positive property.
- **Theorem 2:** It is possible that there exists a God-like object x (in at least one possible world, there exists a God-like object x).
- **Definition 2:** A property φ is an essential property of an object x if, x has property φ, and every property ψ of x necessarily (in all possible worlds) and generally (for all objects) follows from φ.
- **Axiom 4:** If a property φ is positive, then it is necessarily positive (positive in all possible worlds).
- **Theorem 3:** If x is God-like, then being God-like is an essential property of x.
- **Definition 3:** An object x "exists necessarily" if each of its essential properties φ applies, in each possible world, to some object y.
- **Axiom 5:** "Necessary existence" is a positive property.
- **Theorem 4:** It is necessarily true (true in all possible worlds) that a God-like object exists.

Aristotle says potential infinity is possible (i.e. a person could theoretically keep counting or dividing forever) but that an actual infinity is impossible.

Anything that is actual has finitude. If something exists, it has a definition. It has boundaries that mark it off from everything else. An actual infinity is therefore a kind of contradiction in terms (and, in Aristotle's view, leads to various absurdities).

This was accepted by most mathematicians; as late as the 1830s, Gauss said, "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking."

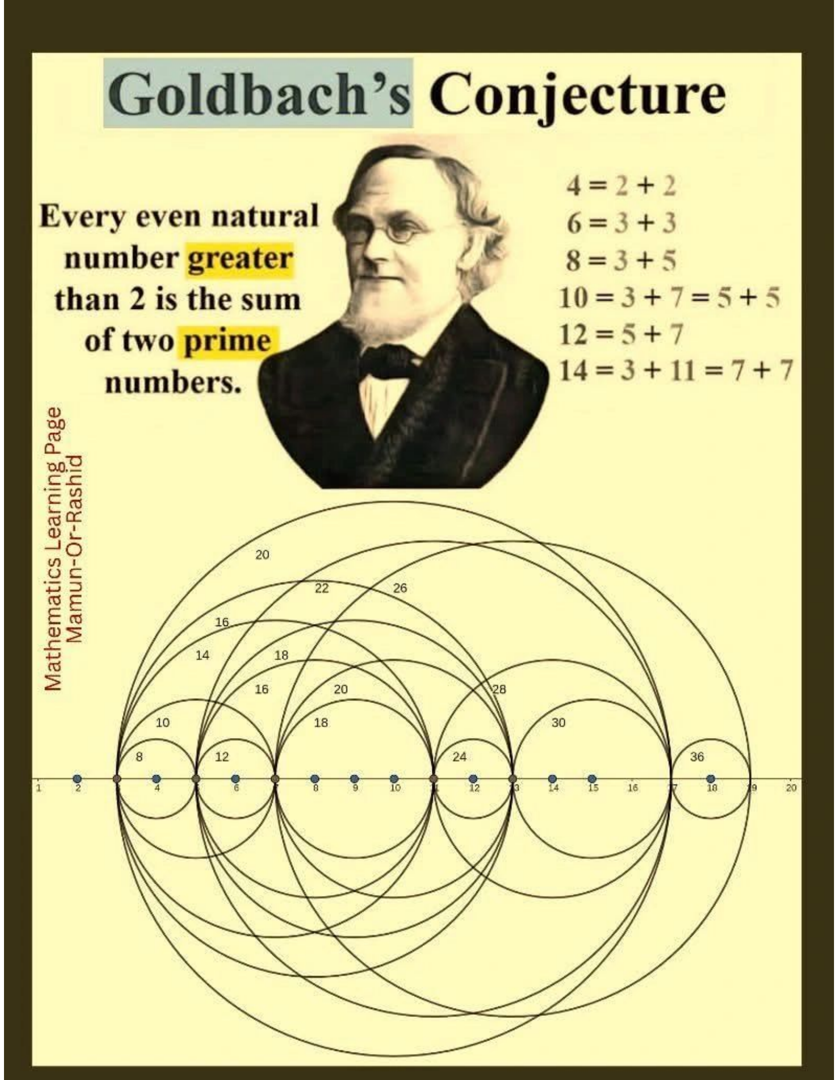
The exceptions were those who were also Christian philosophers and theologians, like Pascal (who noted that infinity was incomprehensible and yet must exist and compared this to the existence of God) and Leibniz (whose invention of calculus uses the concept of infinitesimals and who believed that an actual infinity of monads made up the universe).

Augustine, City of God, Book 12, Chapter 18

“The infinity of number, though there be no numbering of infinite numbers, is yet not incomprehensible by Him whose understanding is infinite. And thus, if everything which is comprehended is defined or made finite by the comprehension of him who knows it, then all infinity is in some ineffable way made finite to God, for **it is comprehensible by His knowledge**. Wherefore, if the infinity of numbers cannot be infinite to the knowledge of God, by which it is comprehended, what are we poor creatures that we should presume to fix limits to His knowledge, and say that unless the same temporal thing be repeated by the same periodic revolutions, God cannot either foreknow His creatures that He may make them, or know them when He has made them? God, whose knowledge is simply manifold, and uniform in its variety, comprehends all incomprehensibles with so incomprehensible a comprehension.”

The consensus seems to be that Goldbach’s conjecture is **true, but not provable.**

It is probably an example of Gödel incompleteness.



Wigner on the Unreasonable Effectiveness of Mathematics

"The great mathematician fully, almost ruthlessly, exploits the domain of permissible reasoning and skirts the impermissible. That his recklessness does not lead him into a morass of contradictions is a miracle in itself: certainly it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess…It is, as Schrodinger has remarked, a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered…It is difficult to avoid the impression that **a miracle confronts us here**, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind's capacity to divine them…The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be **grateful** for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."